Integration

Indefinite Integrals

A function $F(x)$ is an antiderivative of a function $f(x)$ if

$$F(x) = f(x)$$

for all x in the domain of f.

$$\therefore F(x) = \int f(x)\, dx$$

where the symbol $\int$ is an integral sign.

- We have to find one antiderivative F of a function f, the other antiderivatives differ from it by a constant.

$$\boxed{\int f(x)\, dx = F(x) + C}$$

where the constant C is the constant of integration.

Example: Evaluate $\int 2x\, dx$

the constant of integration

Solution: $\int 2x\, dx = x^2 + C$

an antiderivative of $2x$

The formula $x^2 + C$ is all the antiderivatives of the

function $2x$.
The functions x^2+1, $x^2-\pi$, and $x^2+\sqrt{2}$ are all antiderivatives of the function $2x$.

Many of the indefinite integrals are found by reversing derivative formulas. as in the following table.

Indefinite integral	Reversed derivative formula
1. $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + c, \; n \neq -1$	$\dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1}\right) = x^n$
2. $\int dx = \int 1 \, dx = x + c$	$\dfrac{d}{dx}(x) = 1$
. $\int \sin kx \, dx = \dfrac{-\cos kx}{k} + c$	$\dfrac{d}{dx}\left(\dfrac{-\cos kx}{k}\right) = \sin kx$
1. $\int \sec^2 x \, dx = \tan x + c$	$\dfrac{d}{dx} \tan x = \sec^2 x$
: $\int \csc^2 x \, dx = -\cot x + c$	$\dfrac{d}{dx}(-\cot x) = \csc^2 x$
i. $\int \sec x \tan x \, dx = \sec x + c$	$\dfrac{d}{dx} \sec x = \sec x \tan x$
L. $\int \csc x \cot x \, dx = -\csc x + c$	$\dfrac{d}{dx}(-\csc x) = \csc x \cot x$

Example: Evaluate the following:

(a) $\int x^5 \, dx = \dfrac{x^6}{6} + c$

(b) $\int \dfrac{1}{\sqrt{x}} \, dx = \int x^{-1/2} \, dx = 2x^{1/2} + c = 2\sqrt{x} + c$

ⓒ $\int \sin 2x \, dx = \frac{-\cos 2x}{2} + c$

ⓓ $\int \cos \frac{x}{2} \, dx = \int \cos\left(\frac{1}{2}x\right) dx = \frac{\sin(1/2)x}{1/2} + c$

$$= 2\sin\frac{x}{2} + c.$$

Initial Value Problems

The initial value problems is the problem of finding a function y of x when we know its derivative and its value y_0 at a particular point x_0.

Example: Find the curve whose slope at the point (x, y) is $3x^2$, if the curve is required to pass through the point $(1, -1)$.

Solution: The differential equation: $\frac{dy}{dx} = 3x^2$

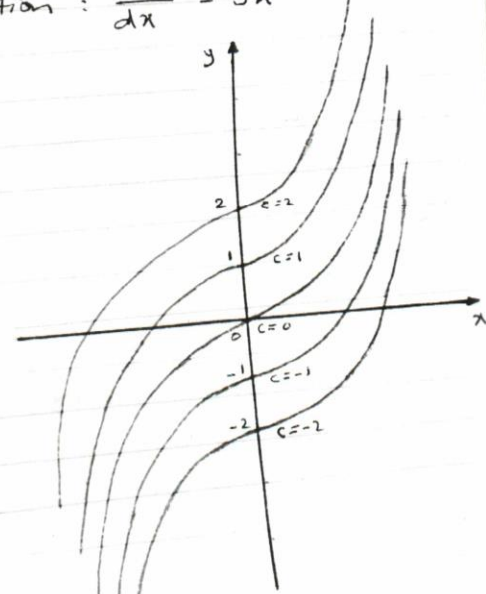
The initial condition: $y(1) = -1$

1- Solve the differential equation:

$$\frac{dy}{dx} = 3x^2$$

$$\int \frac{dy}{dx} \, dx = \int 3x^2 \, dx$$

$$y = x^3 + C$$



2- Evaluate c:

$$y = x^3 + c$$
$$-1 = (1)^3 + c$$
$$\therefore c = -2$$

The curve we want is $y = x^3 - 2$

Example: Find the curve whose slope is $2x$ and $y = 4$ when $x = 1$.

Solution: The differential equation: $\frac{dy}{dx} = 2x$

The initial condition $y(1) = 4$

1- Solve the differential equation:

$$\frac{dy}{dx} = 2x$$
$$y = x^2 + c$$

2- Evaluate c:

$$y = x^2 + c$$
$$4 = (1) + c$$
$$\therefore c = 3$$

The curve is $y = x^2 + 3$

Mathematical Modeling

Example: A balloon ascending at the rate of 12 ft/sec is at height 80 ft above the ground when a package is dropped. How long does it take the package to reach the ground?

Solution:

Let $V_{(t)}$ is the velocity of the package at t.
$S_{(t)}$ is the height of the package above the ground.

The acceleration of gravity near the surface of the earth is 32 ft/sec^2.

$$\therefore \quad a = \frac{dv}{dt} = -32$$

$\therefore$ Differential equation : $\frac{dv}{dt} = -32$

Initial condition : $V(0) = 12$

1- Solve the differential equation:

$$\frac{dv}{dt} = -32$$

$$\int \frac{dv}{dt}\, dt = \int -32\, dt$$

$$v = -32t + c$$

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2- Evaluate c:

$$12 = -32(0) + c$$
$$c = 12$$

$\therefore$ The Solution of the initial value problem is:

$$v = -32t + 12$$

Since velocity is the derivative of height and the height of the package is 80 ft at the time $t=0$ when it is dropped, now we have a second initial value

Differential equation: $\frac{ds}{dt} = -32t + 12$

Initial condition: $s(0) = 80$

Solve this initial value proplem to find the height as a function of t.

$$\frac{ds}{dt} = -32t + 12$$

$$\int \frac{ds}{dt}\, dt = \int (-32t + 12)\, dt$$

$$s = -16t^2 + 12t + c$$

Evaluate c:

$$80 = -16(0)^2 + 12(0) + c$$

$$\therefore c = 80$$

The package's height above ground at time t is

$$S = -16t^2 + 12t + 80$$

Use the solution to find how long it takes the package to reach the ground, we set S equal to 0 and solve for t:

$$-16t^2 + 12t + 80 = 0$$
$$-4t^2 + 3t + 20 = 0$$

$$t = -1.89 \quad \text{or} \quad t = 2.64$$

$\therefore$ The package hits the ground about 2.64 sec after it is dropped from the balloon.

Problems

1- Find an antiderivative for each function, Check your answers by differentiation.

(a) $6x$ (b) x^7 (c) $x^7 - 6x + 8$ (d) $-3x^{-4}$

(e) x^{-4} (f) $x^3 - \frac{1}{x^2}$ (g) $\sqrt{x} + \frac{1}{\sqrt{x}}$ (h) $-\frac{1}{3}x^{-4/3}$

(i) $\sec^2 x$ (j) $\frac{2}{3}\sec^2\frac{x}{3}$ (k) $-\sec^2\frac{3x}{2}$ (l) $4\sec 3x \tan 3x$

2- Evaluate the integrals, check your answers by differentiation:

ⓐ $\int (x+1)\,dx$ ⓑ $\int \left(3t^2 + \frac{t}{2}\right) dt$ ⓒ $\int x^{-1/3}\,dx$

ⓓ $\int (2x^3 - 5x + 7)\,dx$ ⓔ $\int (\sqrt{x} + \sqrt[3]{x})\,dx$ ⓕ $7\sin\frac{\theta}{3}\,d\theta$

ⓖ $\int \cot^2 x\,dx$ ⓗ $\int 2x(1 - x^{-3})\,dx$ ⓘ $\int \left(8y - \frac{2}{y^{1/4}}\right) dy$

3- Find the curve whose slope $-x$ and $y = 1$ when $x = -1$

4- Find the body's position at time t

ⓐ $v = 9.8t + 5$, $S(0) = 10$

ⓑ $v = \frac{2}{\pi} \cos \frac{2t}{\pi}$, $S(\pi^2) = 1$

5- Find the body's position at time t.

ⓐ $a = 32$, $V(0) = 20$, $S(0) = 5$

ⓑ $a = -4 \sin 2t$, $V(0) = 2$, $S(0) = -3$.

6- Solve the initial value problems

ⓐ $\frac{dy}{dx} = \frac{1}{x^2} + x$, $y(2) = 1$. ⓑ $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, $y(4) = 0$

ⓒ $\frac{ds}{dt} = \cos t + \sin t$, $S(\pi) = 1$ ⓓ $y^{(4)} = -\sin t + \cos t$; $y'''(0) = 7$

$y''(0) = y'(0) = -1$, $y(0) = 0$

Integration by Substitution

Rules for indefinite integration

1- Costant Multiple Rule: $\int k\, f(x)\, dx = k \int f(x)\, dx$

2- Rule for Negatives: $\int -f(x)\, dx = -\int f(x)\, dx$

3- Sum and Difference Rule: $\int [f(x) \mp g(x)]\, dx = \int f(x)\, dx \mp \int g(x)\, dx$

Example: Evaluate $\int 5 \sec x \tan x\, dx$

Solution:

$$5 \int \sec x \tan x\, dx$$

$$= 5 \sec x + c$$

Example: Evaluate $\int (x^2 - 2x + 5)\, dx$

Solution

$$= \int x^2\, dx - \int 2x\, dx + \int 5\, dx$$

$$= \frac{x^3}{3} - x^2 + 5x + c$$

Example: Integrating $\sin^2 x$ and $\cos^2 x$

ⓐ $\int \sin^2 x\, dx$

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$\because \sin^2 x = \frac{1-\cos 2x}{2}$

$\therefore \int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx$

$= \frac{1}{2}\int (1-\cos 2x)\,dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$

$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$

(b) $\int \cos^2 x\,dx$

$\because \cos^2 x = \frac{1+\cos 2x}{2}$

$\therefore \int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx$

$= \frac{1}{2}\int (1+\cos 2x)\,dx$

$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\,dx$

$= \frac{1}{2}x + \frac{1}{2}\frac{\sin 2x}{2} + C$

$= \frac{x}{2} + \frac{\sin 2x}{4} + C$

The Power Rule in Integral Form

If u is any differentiable function, then

$$\boxed{\int u^n \, du = \frac{u^{n+1}}{n+1} + C}$$

Example: Evaluate $\int \sqrt{1+y^2} \cdot 2y \, dy$

Solution: let $u = 1+y^2$

$du = 2y \, dy$

$$\therefore \int u^{1/2} \, du$$

$$= \frac{u^{1/2+1}}{1/2+1} + C = \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{3} \left(1+y^2\right)^{3/2} + C$$

Example: Evaluate $\int \sqrt{4t-1} \, dt$

Solution: let $u = 4t-1$, $du = 4 \, dt \Rightarrow \frac{1}{4} du = dt$

$$= \int u^{1/2} \cdot \frac{1}{4} \, du$$

$$= \frac{1}{4} \int u^{1/2} \, du \quad = \frac{1}{4} \frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{6} u^{3/2} + c$$

$$= \frac{1}{6} (4t - 1)^{3/2} + c$$

The Substitution Method of Integration

If $\int f(g(x)) \cdot g'(x)\, dx$

1- Substitute $u = g(x)$, $du = g'(x)\, dx$

$$\int f(u)\, du$$

2- Evaluate by finding an antiderivative

$$F(u) + c$$

3- Replace u by $g(x)$

$$F(g(x)) + c$$

Example: Evaluate $\int \cos(7\theta + 5)\, d\theta$

Solution: let $u = 7\theta + 5$

$du = 7\, d\theta \Rightarrow \frac{1}{7} du = d\theta$

$$= \int \cos u \cdot \frac{1}{7}\, du$$

$$= \frac{1}{7} \int \cos u \, du$$

$$= \frac{1}{7} \sin u + c \quad = \frac{1}{7} \sin(7\theta + 5) + C$$

Example: Evaluate $\int x^2 \sin x^3 \, dx$

Solution: $= \int \sin x^3 \cdot x^2 \, dx$

let $u = x^3$

$du = 3x^2\, dx \Rightarrow \frac{1}{3}\, du = x^2\, dx$

$$= \int \sin u \cdot \frac{1}{3}\, du$$

$$= \frac{1}{3} \int \sin u \, du$$

$$= \frac{1}{3}(-\cos u) + c \quad = \frac{-1}{3} \cos x^3 + C$$

Example: Evaluate $\int \frac{1}{\cos^2 2x}\, dx$

Solution:

$$\int \frac{1}{\cos^2 2x}\, dx = \int \sec^2 2x \, dx$$

let $u = 2x$

$du = 2\, dx \Rightarrow \frac{1}{2}\, du = dx$

$$= \int \sec^2 u \cdot \frac{1}{2}\, du$$

$$= \frac{1}{2} \int \sec^2 u \; du$$

$$= \frac{1}{2} \tan u + c = \frac{1}{2} \tan 2x + c$$

Example: Evaluate $\displaystyle\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$

Solution: ① Substitute $u = z^2 + 1$

$du = 2z \; dz$

$$= \int \frac{du}{u^{1/3}}$$

$$= \int u^{-1/3}\, du = \frac{u^{2/3}}{2/3} + c$$

$$= \frac{3}{2} u^{2/3} + c = \frac{3}{2} (z^2+1)^{2/3} + c$$

② Substitute $u = \sqrt[3]{z^2+1}$

$$u^3 = (z^2+1)$$

$$3u^2\, du = 2z \; dz$$

$$= \int \frac{3u^2\, du}{u} = 3 \int u \, du$$

$$= 3\, \frac{u^2}{2} + c$$

$$= \frac{3}{2} (z^2+1)^{2/3} + c$$

Example : Solve the initial value problem

$$\frac{ds}{dt} = 12t\,(3t^2-1)^3 \quad , \quad S(1) = 3$$

Solution :

let $u = 3t^2 - 1$

$du = 6t\ dt$

$2\,du = 12t\ dt$

$$ds = \int u^3 \cdot 2\ du$$

$$S = \frac{u^4}{4} \cdot 2 + c$$

$$S = \frac{1}{2} u^4 + c$$

$$S = \frac{1}{2}(3t^2-1)^4 + c$$

$$S(1) = 3$$

$$3 = \frac{1}{2}(3(1)^2-1)^4 + c \Rightarrow c = -5$$

$$\therefore\ S = \frac{1}{2}(3t^2-1)^4 - 5$$

Problems

1- Evaluate the indefinite integrals

ⓐ $\int x \sin(2x^2)\, dx \quad , \quad u = 2x^2$

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(b) $\int (1-\cos\frac{t}{2})^2 \sin\frac{t}{2}\, dt$, $u = 1-\cos\frac{t}{2}$

(c) $\int 28(7x-2)^{-5}\, dx$, $u = 7x-2$

(d) $\int x^3 (x^4-1)^2\, dx$, $u = x^4-1$

(e) $\int \frac{9r^2\, dr}{\sqrt{1-r^3}}$, $u = 1-r^3$

(g) $\int \csc^2 2\theta \cot 2\theta\, d\theta$, → ① $u = \cot 2\theta$ → ② $u = \csc 2\theta$

(f) $\int \frac{dx}{\sqrt{5x+8}}$, → ① $u = 5x+8$ → ② $u = \sqrt{5x+8}$

2- Evaluate the integrals

(a) $\int \sqrt{3-2s}\, ds$, (b) $\int \sec^2(3x+2)\, dx$

(c) $\int \sin^5\frac{x}{3} \cos\frac{x}{3}\, dx$, (d) $\int \tan^7\frac{x}{2} \sec^2\frac{x}{2}\, dx$

3- Solve the initial value Problems

(a) $\frac{dy}{dx} = 4x(x^2+8)^{-1/3}$, $y(0) = 0$

(b) $\frac{ds}{dt} = 8\sin^2(t+\frac{\pi}{12})$, $s(0) = 8$